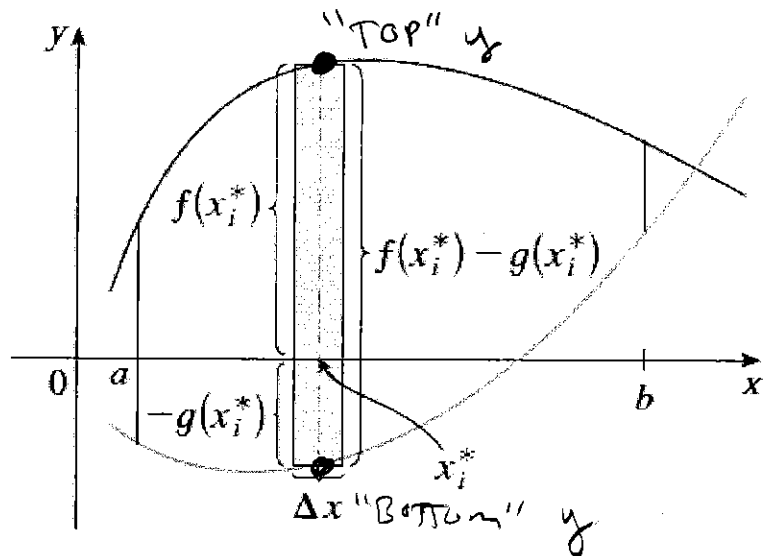
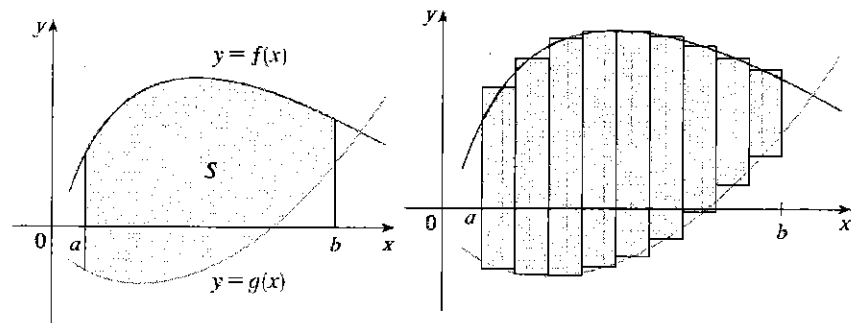


Ch 6: Basic Integral Applications

6.1 Areas Between Curves

Using dx:

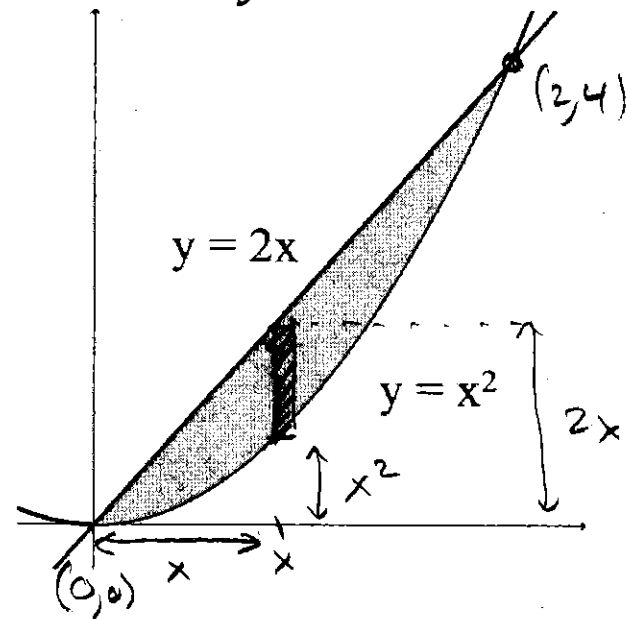


(a) Typical rectangle

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n (f(x_i) - g(x_i)) \Delta x$$

Example: Find the area bounded between $y = 2x$ and $y = x^2$.

$$\begin{aligned} 2x &= x^2 \\ \Rightarrow 0 &= x^2 - 2x \\ 0 &= x(x-2) \end{aligned}$$



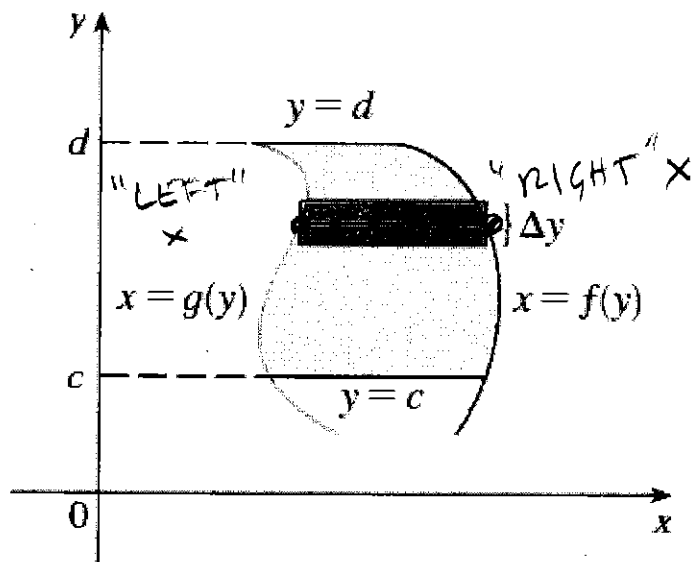
$$\int_0^2 2x - x^2 dx$$

$$x^2 - \frac{1}{3}x^3 \Big|_0^2 = (2^2 - \frac{1}{3}(2)^3) - (0^2 - \frac{1}{3}(0)^3)$$

$$= 4 - \frac{8}{3}$$

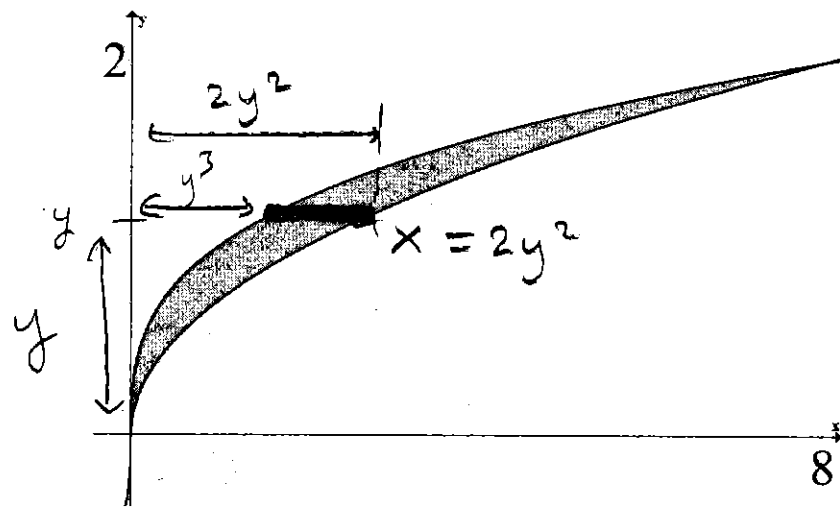
$$= \boxed{\frac{4}{3}}$$

Using dy :



$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n (f(y_i) - g(y_i)) \Delta y$$

Example: Set up an integral for the area bounded between $x = 2y^2$ and $x = y^3$ (shown below) using dy .



$$\begin{aligned} & \int_0^2 (2y^2 - y^3) dy \\ &= \left. \frac{2}{3}y^3 - \frac{1}{4}y^4 \right|_0^2 \\ &= \left(\frac{2}{3}(2)^3 - \frac{1}{4}(2)^4 \right) - 0 \\ &= \frac{16}{3} - 4 = \boxed{\frac{4}{3}} \end{aligned}$$

Summary: The area between curves

1. Draw picture finding all intersections.
2. Choose dx or dy. Get **everything** in terms of the variable you choose.
3. Draw a typical approx. rectangle.
4. Set up as follows:

$$\text{Area} = \int_a^b (\text{TOP} - \text{BOTTOM}) dx$$

$$\text{Area} = \int_c^d (\text{RIGHT} - \text{LEFT}) dy$$

HERE IS WHAT IT WOULD LOOK LIKE

$$\int_0^1 \sqrt{x} - (-\sqrt{x}) dx + \int_1^4 \sqrt{x} - (x-2) dx$$

ALSO CORRECT

Example: Set up an integral (or integrals) that give the area of the region bounded by $x = y^2$ and $y = x - 2$.

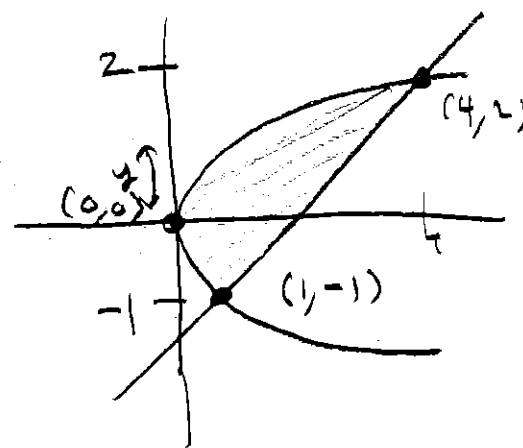
$$x = y^2 \Leftrightarrow \begin{cases} y = \sqrt{x} \\ y = -\sqrt{x} \end{cases}$$

$$x = y + 2 \Leftrightarrow y = x - 2$$

$$y^2 = y + 2$$

$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$



DON'T USE X! $\left\{ \begin{array}{l} \text{BOTTOM CHANGES} \\ \text{AT } x=1 \end{array} \right.$

$$\int_{-1}^2 \text{RIGHT} - \text{LEFT} dy$$

$$\int_{-1}^2 y + 2 - y^2 dy$$

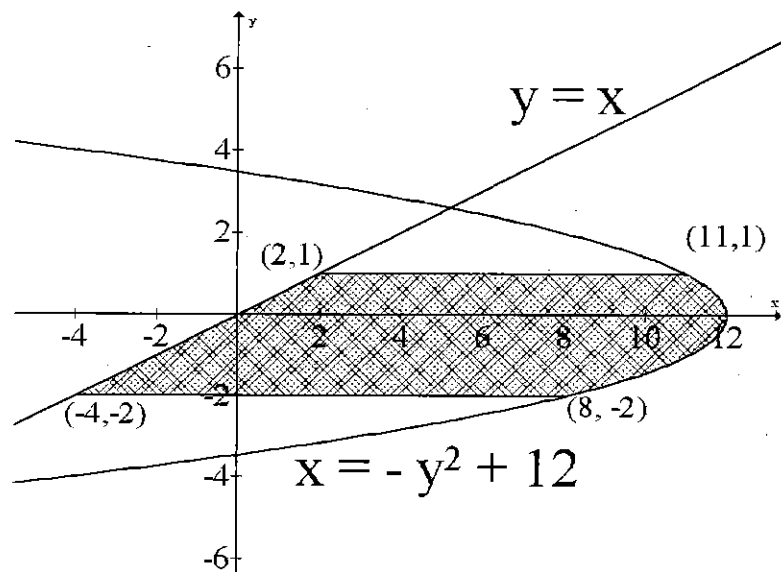
$$\left. \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \right|_{-1}^2$$

$$\left(\frac{1}{2}(2)^2 + 2(2) - \frac{1}{3}(2)^3 \right) - \left(\frac{1}{2}(-1)^2 + 2(-1) - \frac{1}{3}(-1)^3 \right)$$

$$\left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right)$$

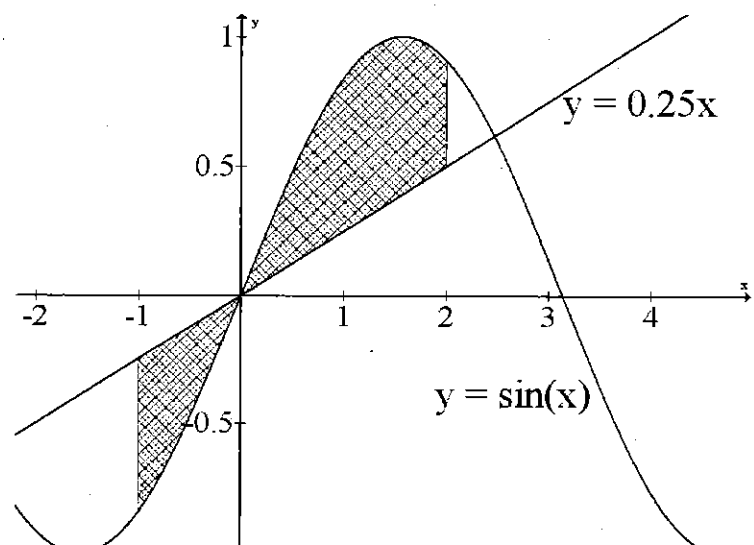
$$8 - \frac{8}{3} - \frac{1}{2} = 5 - \frac{1}{2} = \boxed{\frac{9}{2}}$$

Set up an integral for the total positive area of the following regions:



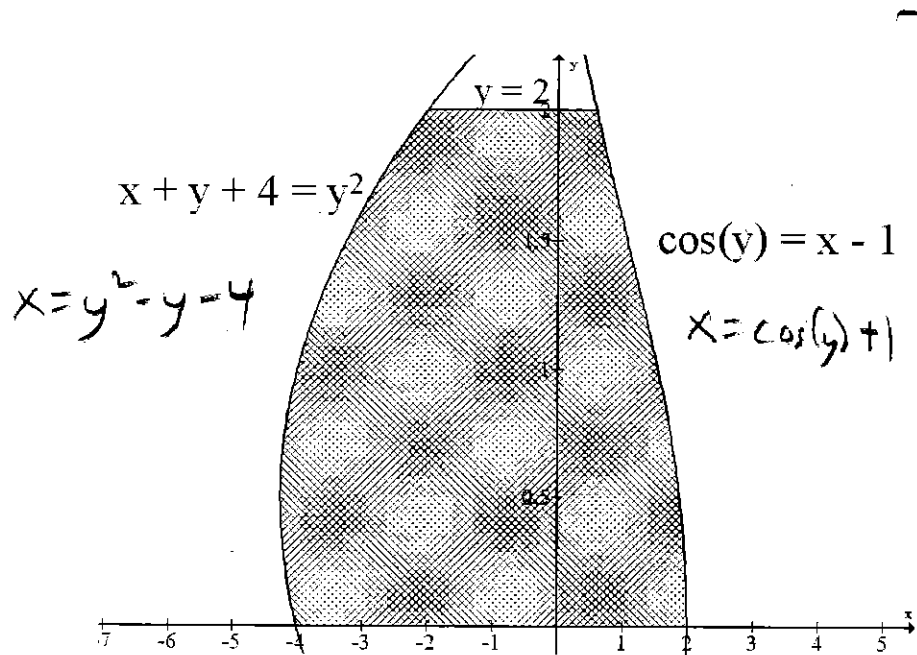
USE $dy!!!$

$$\int_{-2}^2 (-y^2 + 12) - y \, dy$$



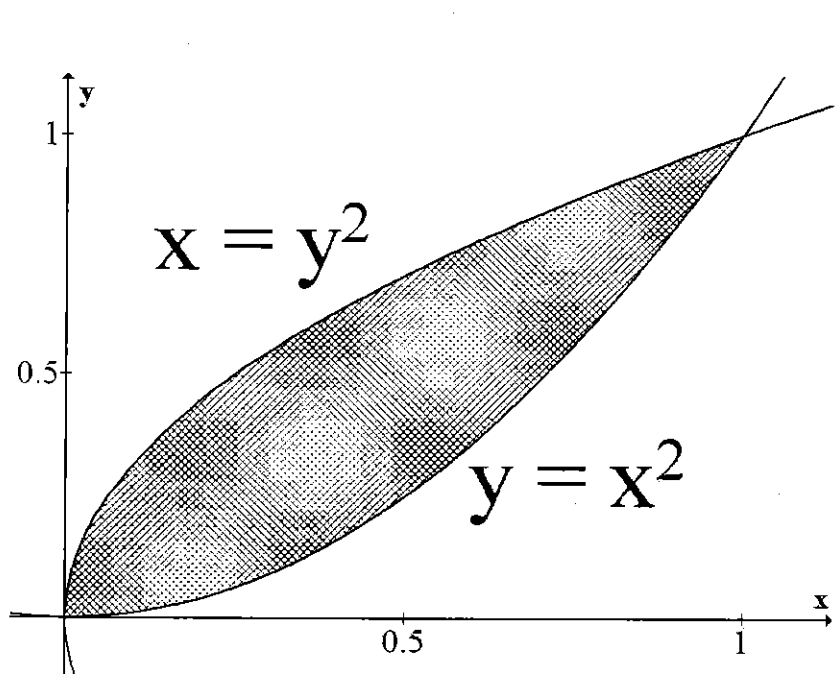
USE $dx!!!$

$$\int_{-1}^0 \frac{1}{4}x - \sin(x) \, dx + \int_0^2 \sin(x) - \frac{1}{4}x \, dx$$



USE
dy!!!

$$\int_0^2 (\cos(y) + 1) - (y^2 - y - 4) dy$$

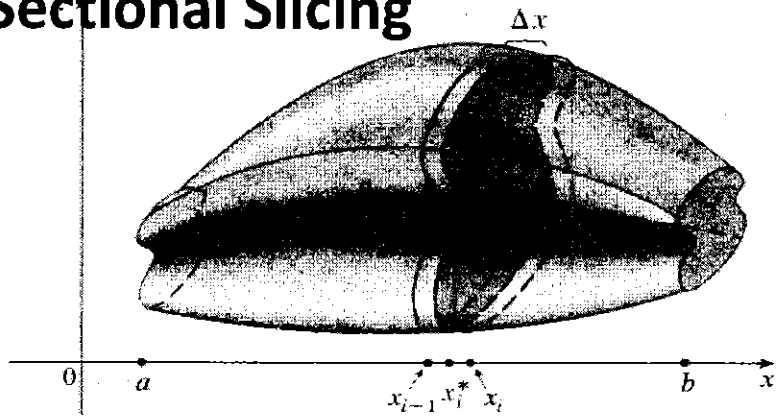


BOTH
work
w/ w

$$dx: \int_0^1 \sqrt{x} - x^2 dx$$

$$dy: \int_0^1 \sqrt{y} - y^2 dy$$

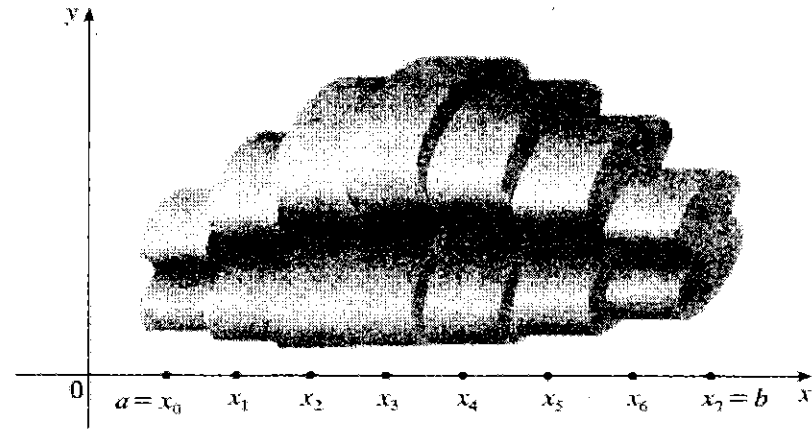
5.2 Finding Volumes Using Cross-Sectional Slicing



If we can find the general formula, $A(x_i)$, for the area of a cross-sectional slice, then we can approximate volume by:

Volume of one slice $\approx A(x_i) \Delta x$

Total Volume $\approx \sum_{i=1}^n A(x_i) \Delta x$



This approximation gets better and better with more subdivisions, so

$$\text{Exact Volume} = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i) \Delta x$$

We conclude


$$\text{Volume} = \int_a^b A(x) dx =$$


$$\int_a^b \text{"Cross-sectional area formula"} dx$$


Volume using cross-sectional slicing

1. Draw region. Cut **perpendicular** to rotation axis. Label x if that cut crosses the x -axis (and y if y -axis). Label **everything** in terms this variable.

2. Formula for cross-sectional area?

disc: Area = $\pi(\text{radius})^2$ 

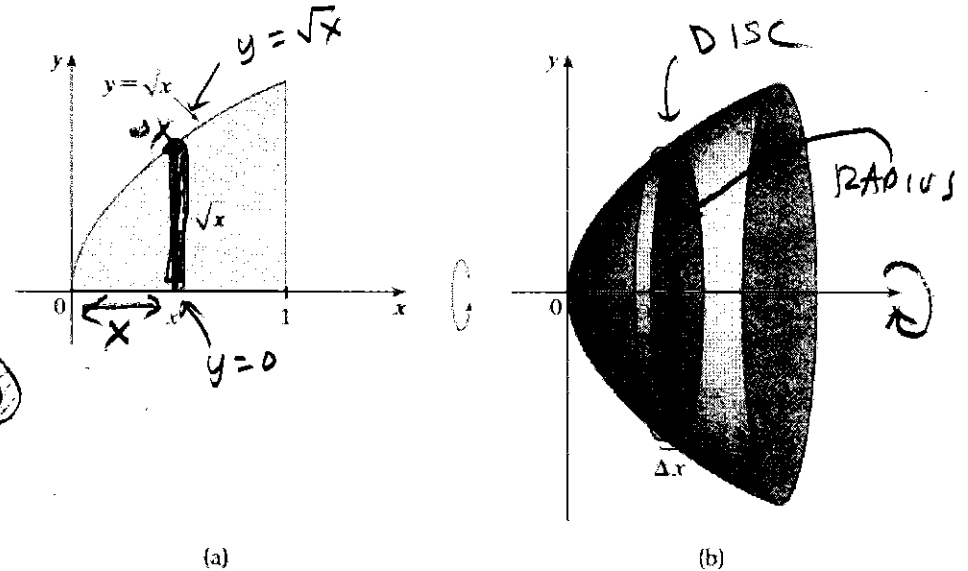
washer: Area = $\pi(\text{outer})^2 - \pi(\text{inner})^2$ 

square: Area = (Height)(Length) 

triangle: Area = $\frac{1}{2}$ (Height)(Length)

3. Integrate the area formula.

Example: Consider the region, R , bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$. Find the volume of the solid obtained by rotating R about the **x -axis**.



$$\begin{aligned} & \int_0^1 \pi (\text{RADIUS})^2 dx \\ &= \int_0^1 \pi (\sqrt{x})^2 dx \\ &= \pi \int_0^1 x dx = \pi \left[\frac{1}{2} x^2 \right]_0^1 \\ &= \boxed{\frac{\pi}{2}} \end{aligned}$$

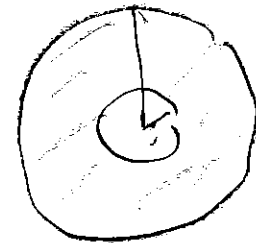
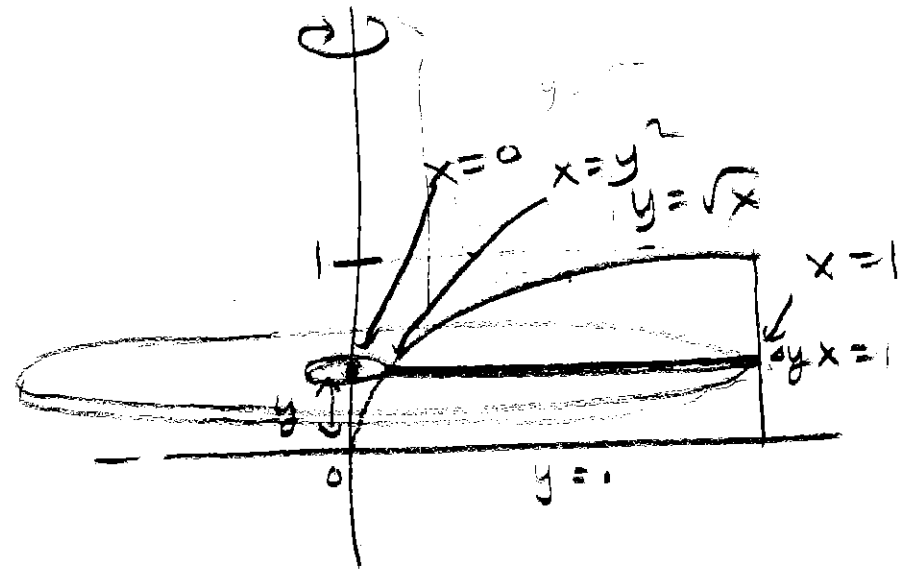
Example: Consider the region, R , bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$. Find the volume of the solid obtained by rotating R about the **y -axis**.

$$\int_0^1 \pi (1)^2 - \pi (y^2)^2 dy$$

$$= \pi \int_0^1 1 - y^4 dy$$

$$= \pi \left(y - \frac{1}{5} y^5 \right) \Big|_0^1$$

$$= \pi \left(1 - \frac{1}{5} \right) = \boxed{\frac{4\pi}{5}}$$



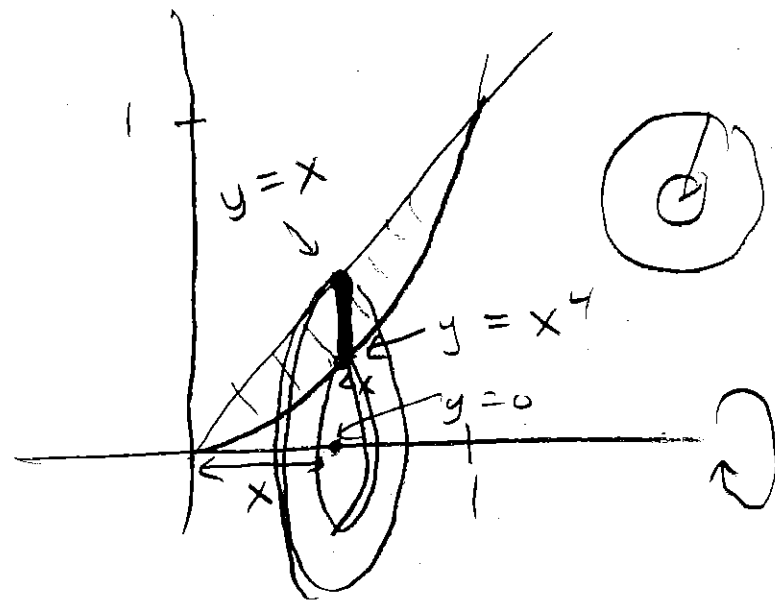
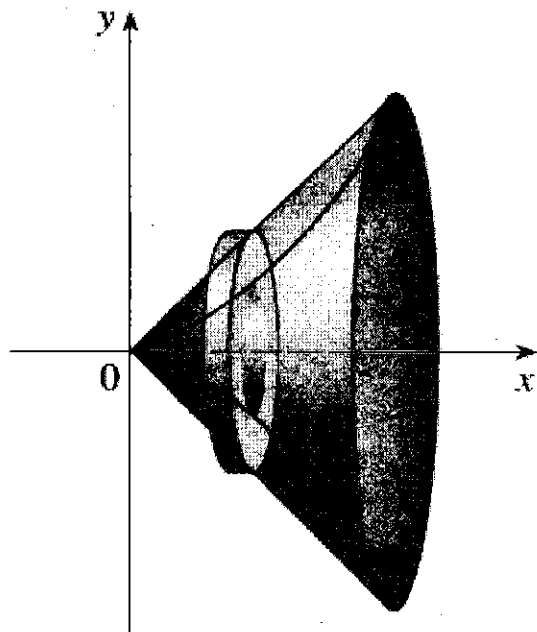
Example: Consider the region, R, bounded by $y = x$ and $y = x^4$. Find the volume of the solid obtained by rotating R about the **x-axis**.

$$\int_0^1 \pi (x)^2 - \pi (x^4)^2 dx$$

$$\pi \int_0^1 x^2 - x^8 dx$$

$$\pi \left(\frac{1}{3} x^3 - \frac{1}{9} x^9 \right) \Big|_0^1$$

$$\pi \left(\frac{1}{3} - \frac{1}{9} \right) = \boxed{\frac{2\pi}{9}}$$



Example: Consider the region, R , bounded by $y = x$ and $y = x^4$. R is the same as the last example).

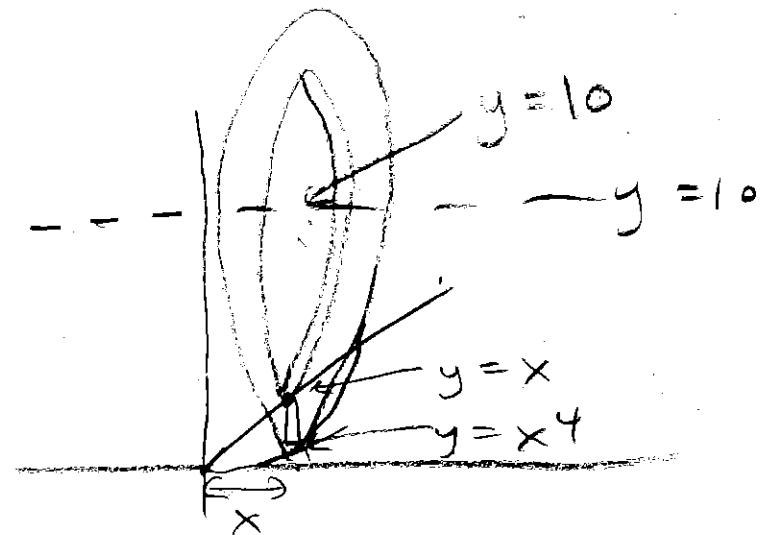
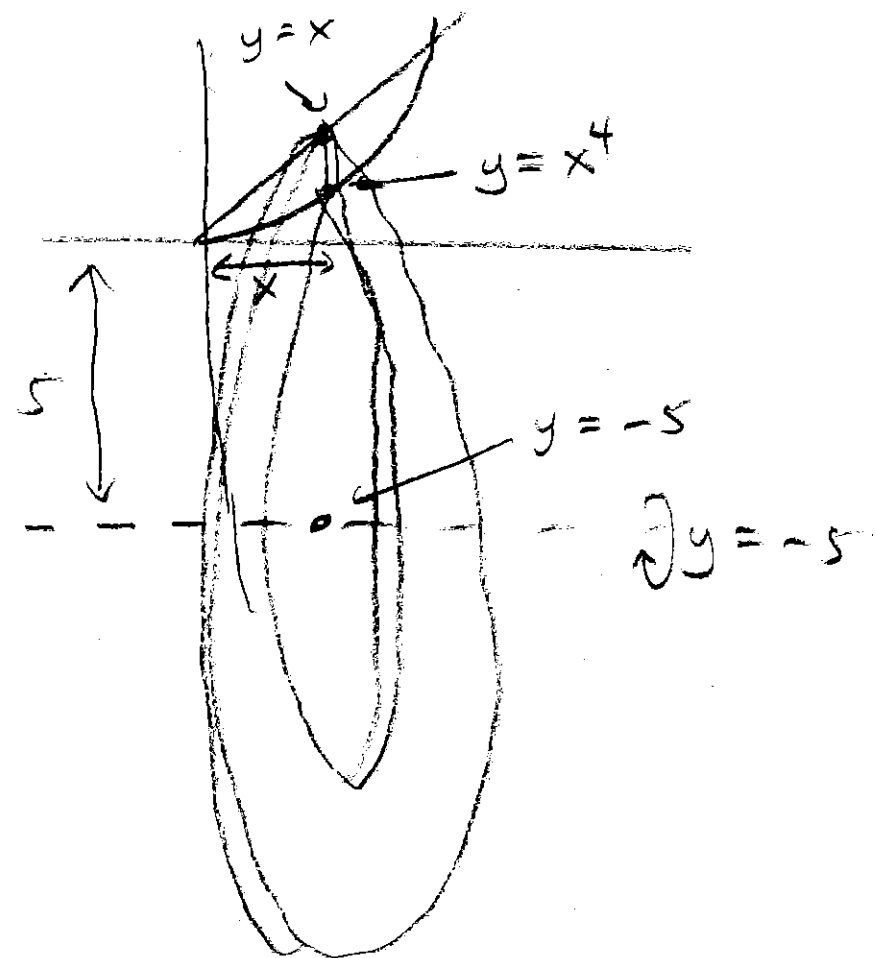
(a) Now rotate about the horizontal line $y = -5$. What changes?

$$\int_0^1 \pi (x - (-5))^2 - \pi (x^4 - (-5))^2 dx$$

$$\pi \int_0^1 (x + 5)^2 - (x^4 + 5)^2 dx$$

(b) Now rotate about the horizontal line $y = 10$. What changes?

$$\int_0^1 \pi (10 - x^4)^2 - \pi (10 - x)^2 dx$$



Example: $y = 2\sqrt{x}$ $x = \left(\frac{y}{2}\right)^{2/3}$
 Consider the region bounded by
 $4x = y^2$ and $y = 2x^3$.

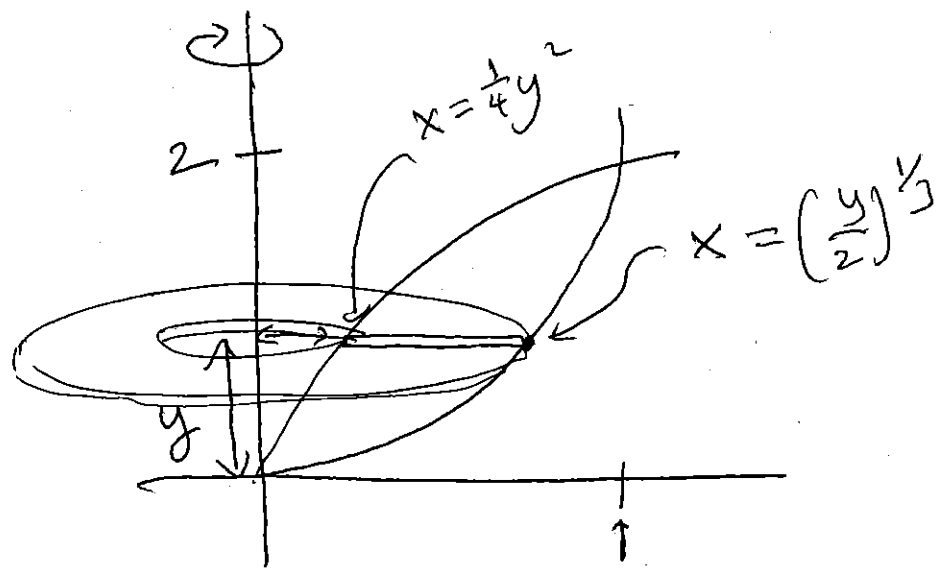
Find the volume of the solid obtained
 by rotating this region about the
y-axis.

STEP 1
 INTERSECTION:

$$4x = (2x^3)^2$$

$$4x = 4x^6 \Rightarrow x=1 \text{ or } x=0$$

\Downarrow \Downarrow
 $y=2$ $y=0$



STEP 2

$$\int_0^2 \pi (\quad)^2 - \pi (\quad)^2 dy$$

STEP 3 Fill in

\uparrow \uparrow
 $\left(\frac{y}{2}\right)^{2/3}$ $\frac{1}{4}y^2$

$$\Rightarrow \pi \int_0^2 \left(\frac{y}{2}\right)^{2/3} - \left(\frac{1}{4}y^2\right)^2 dy$$

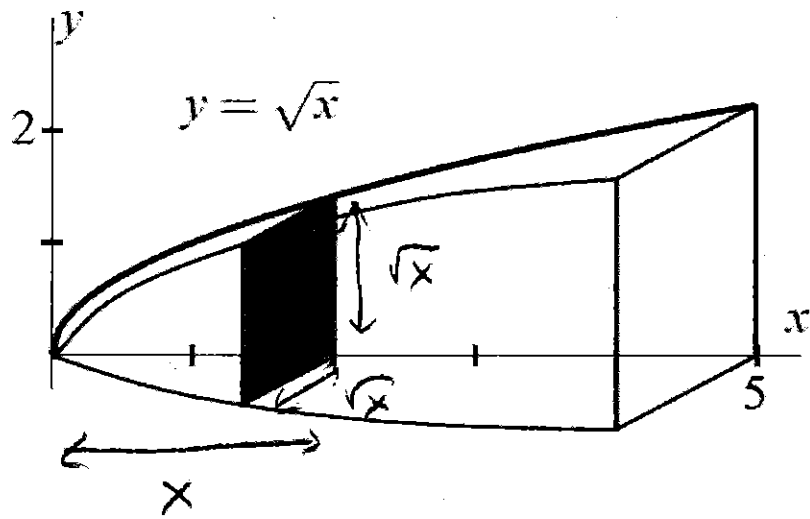
$$\pi \int_0^2 \frac{y^{2/3}}{2^{2/3}} - \frac{1}{16} y^4 dy$$

$$\pi \left[\frac{1}{2^{2/3}} \frac{3}{5} y^{5/3} - \frac{1}{16} \frac{1}{5} y^5 \right]_0^2$$

$$= \frac{4}{5} \pi \approx 2.5133$$

Example:

From an old final and homework)
Find the volume of the solid shown.
The cross-sections are squares.



$$\int_0^5 (\text{HEIGHT})(\text{LENGTH}) dx$$

$$\int_0^5 \sqrt{x} \cdot \sqrt{x} dx$$

$$\int_0^5 x dx = \frac{1}{2} x^2 \Big|_0^5 = \frac{1}{2} 25 = \boxed{\frac{25}{2}}$$

Summary (Cross-sectional slicing):

1. Draw Label
2. Cross-sectional area?
3. Integrate area.

This method has a major limitation:

5.2 method about x -axis, must use dx .

5.2 method about y -axis, must use dy .

What if the regions is rotated about the x -axis and we need to use dy ?
or about y -axis and we need dx ?)

In these cases, 6.2 “Cross-sectional slicing” wouldn’t work!

We need another method.

That is what we will do in 6.3.

Close Wed: HW_3A,3B,3C

(complete sooner!)

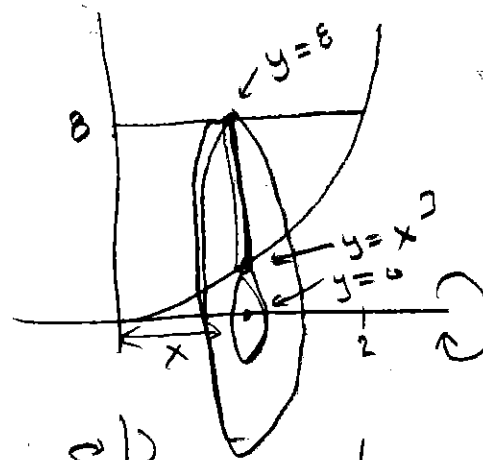
Exam 1 is Thurs (4.9, 5.1-5.5, 6.1-6.3)

Entry Task:

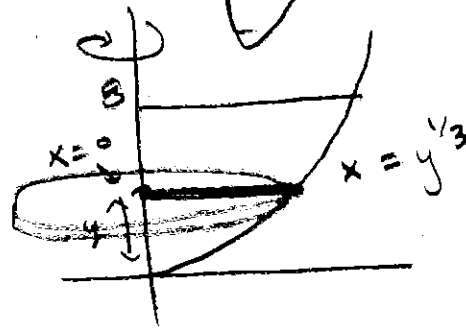
Consider the region R bounded by $y = x^3$, $y = 8$, and $x = 0$.

Set up the integrals that would give the volume of the solid obtained by rotating R about the ...

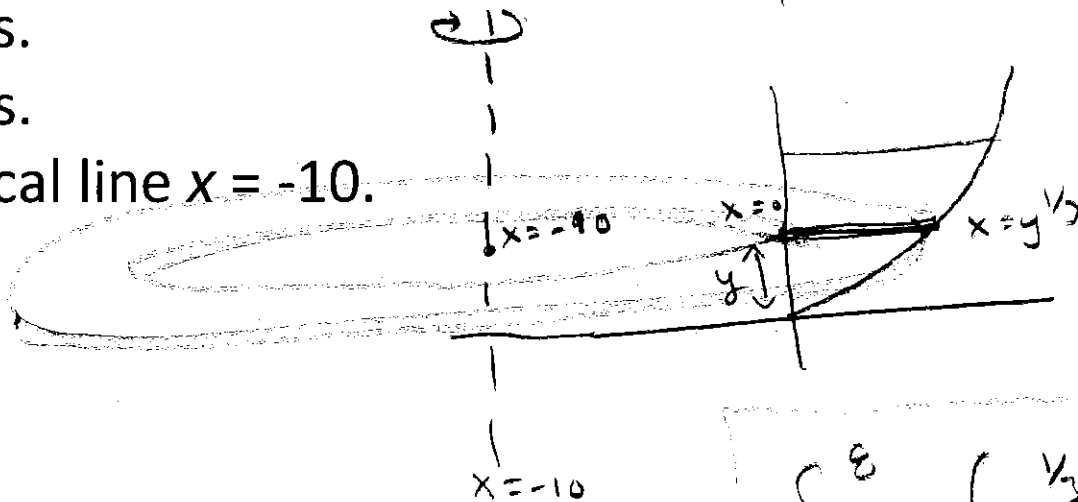
- (a) ... x -axis.
- (b) ... y -axis.
- (c) ... vertical line $x = -10$.



$$\int_0^2 \pi(8^2 - \pi(x^3)^2) dx$$
$$\pi \int_0^2 64 - x^6 dx$$



$$\int_0^8 \pi (y^{1/3})^2 dy$$
$$= \pi \int_0^8 y^{2/3} dy$$



$$\int_0^8 \pi (y^{1/3} + 10)^2 - \pi(10)^2 dy$$